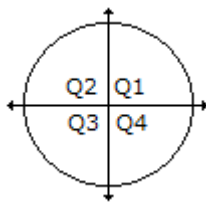
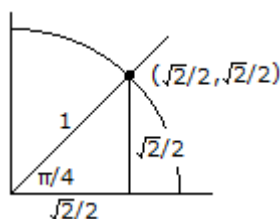


The Unit Circle Exact Measurements and Symmetry

Consider the unit circle: a circle of radius 1, centered at the origin. The x and y -axes break up the plane into four quadrants, labeled 1-4, as shown below:



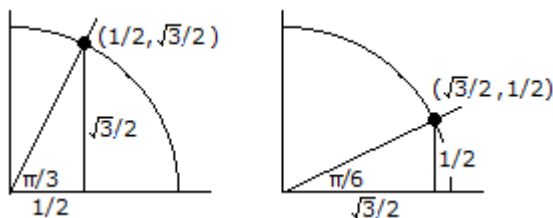
Now look at Quadrant 1. If we sketch in a ray at an angle of $\frac{\pi}{4}$ radians (45 degrees), we can calculate the coordinate (x, y) by using Pythagoras' Theorem to determine the lengths of the legs (remember, the x -coordinate is the length of the horizontal leg, the y -coordinate the length of the vertical leg):



Therefore, we have these exact measurements:

- $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\tan \frac{\pi}{4} = 1$.

We can also sketch rays at angles of $\frac{\pi}{6}$ radians (30 degrees) and $\frac{\pi}{3}$ radians (60 degrees) in Quadrant 1. These form parts of an equilateral (all sides and angles equal) triangle. Using Pythagoras' Theorem, we can determine exact coordinates as shown below:



This gives us more exact measurements:

- $\sin \frac{\pi}{6} = \frac{1}{2}$, $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\tan \frac{\pi}{6} = \frac{\sqrt{3}}{3}$,
- $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos \frac{\pi}{3} = \frac{1}{2}$ and $\tan \frac{\pi}{3} = \sqrt{3}$.

You should memorize these so-called “First Quadrant Exact Measures”. A neat way to memorize these values is shown on the next page:

• **First Quadrant Exact Measures:**

- 1) Set up a table with three columns: angle measures for 0, 30, 45, 60 and 90 degrees and their equivalent radians, and a column for the cosine and sine values of these angles.
- 2) Fill in “blank roots over 2” in the cosine and sine columns.
- 3) Fill in these blank roots with 4, 3, 2, 1, 0 for the cosine, and 0, 1, 2, 3, 4 for the sine.
- 4) Simplify what you can, and you have the table!

θ	$\cos \theta$	$\sin \theta$		θ	$\cos \theta$	$\sin \theta$		θ	$\cos \theta$	$\sin \theta$
0	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{0}}{2}$		0	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{0}}{2}$		0	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{2} = 0$
30 ($\frac{\pi}{6}$)	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2}$	$\Rightarrow$	30 ($\frac{\pi}{6}$)	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2}$	$\Rightarrow$	30 ($\frac{\pi}{6}$)	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$
45 ($\frac{\pi}{4}$)	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$		45 ($\frac{\pi}{4}$)	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$		45 ($\frac{\pi}{4}$)	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60 ($\frac{\pi}{3}$)	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$		60 ($\frac{\pi}{3}$)	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$		60 ($\frac{\pi}{3}$)	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90 ($\frac{\pi}{2}$)	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{4}}{2}$		90 ($\frac{\pi}{2}$)	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{4}}{2}$		90 ($\frac{\pi}{2}$)	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{4}}{2} = 1$

Other Quadrants:

The cosine, sine and tangent values change sign according to the quadrant in which the angle θ is located, according to this table:

Quadrant 2	Quadrant 1
$\cos \theta < 0$	$\cos \theta > 0$
$\sin \theta > 0$	$\sin \theta > 0$
$\tan \theta < 0$	$\tan \theta > 0$
Quadrant 3	Quadrant 4
$\cos \theta < 0$	$\cos \theta > 0$
$\sin \theta < 0$	$\sin \theta < 0$
$\tan \theta > 0$	$\tan \theta < 0$

To determine exact values in other quadrants, use symmetry to “revert” to a reference angle in the first quadrant, then attach a negative sign if necessary.

Examples:

- $\cos \frac{2\pi}{3} = -\frac{1}{2}$ since the angle $\frac{2\pi}{3}$ is in Quadrant 2, where the cosine is negative, and the angle $\frac{2\pi}{3}$ is symmetrical with the angle $\frac{\pi}{3}$ in Quadrant 1. So we look up $\cos \frac{\pi}{3}$ on our table, see that it is $\frac{1}{2}$, then attach the negative sign.
- $\sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}$. The angle $\frac{3\pi}{4}$ is in Quadrant 2, where the sine is positive. The angle $\frac{3\pi}{4}$ is symmetrical with the angle $\frac{\pi}{4}$ in Quadrant 1. We look up $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, and keep the sign positive.

Note: the tangent column can be found by remembering that $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

The Complete Unit Circle with Exact Measurements

Quadrant 1 θ	$\cos \theta$	$\sin \theta$		Quadrant 3 θ	$\cos \theta$	$\sin \theta$
0 (0 deg)	1	0		π (180 deg)	-1	0
$\pi/6$ (30 deg)	$\sqrt{3}/2$	$1/2$		$7\pi/6$ (210 deg)	$-\sqrt{3}/2$	$-1/2$
$\pi/4$ (45 deg)	$\sqrt{2}/2$	$\sqrt{2}/2$		$5\pi/4$ (225 deg)	$-\sqrt{2}/2$	$-\sqrt{2}/2$
$\pi/3$ (60 deg)	$1/2$	$\sqrt{3}/2$		$4\pi/3$ (240 deg)	$-1/2$	$-\sqrt{3}/2$
Quadrant 2 θ				Quadrant 4 θ		
$\pi/2$ (90 deg)	0	1		$3\pi/2$ (270 deg)	0	-1
$2\pi/3$ (120 deg)	$-1/2$	$\sqrt{3}/2$		$5\pi/3$ (300 deg)	$1/2$	$-\sqrt{3}/2$
$3\pi/4$ (135 deg)	$-\sqrt{2}/2$	$\sqrt{2}/2$		$7\pi/4$ (315 deg)	$\sqrt{2}/2$	$-\sqrt{2}/2$
$5\pi/6$ (150 deg)	$-\sqrt{3}/2$	$1/2$		$11\pi/6$ (330 deg)	$\sqrt{3}/2$	$-1/2$

